colorado math circle

Base Fib

Euler Group

September 2016

Ordered Partitions

1. How many sequences of 1s and 2s sum to 10? For example, for the number 3, there are three sequences: $1+1+1$, $1+2$, and $2+1$.

Golden Ratio. Let $\phi = \dfrac{1+\sqrt{5}}{2}$. The powers ϕ^n can be expressed in the form $\phi^n = a\phi + b$ where a and b are integers. For example:

$$\phi^1 = \phi + 0 \qquad \phi^2 = \phi + 1 \qquad \phi^3 = 2\phi + 1$$

2a. Find expressions for a and b in terms of n. Prove that the formula is true for all positive integers n.

2b. There is a similar formula for powers of $\dfrac{1}{\phi}$: $\dfrac{1}{\phi^n} = c \cdot \dfrac{1}{\phi} + d$. Find expressions for c and d.

2c. Use the formulas for ϕ^n and $\left(-\dfrac{1}{\phi}\right)^n$ to find a formula for F_n.

2d. If you apply your formula for F_n to the ratio F_n/F_{n-1}, what happens as n increases?

An Identity.

3a. Derive a formula for $\sum_{i=1}^{n} F_i^2$. Prove that it is true for all n.

3b. Sketch a diagram to illustrate.

Base Fib. Suppose we create a positional number system using the digits 0 and 1 where each column is a Fibonacci number. Then the first few positive integers can be represented as shown below.

n	5	3	2	1
1				1
2			1	0
3		1	0	0
3			1	1
4		1	0	1
5	1	0	0	0
5		1	1	0

4a. Find Fibonacci representations for the numbers 6 to 13 and for the numbers 35 and 60. If a number has more than one representation, list all of them.

4b. If the minimal Fibonacci representation for a number n has the fewest number of 1s, is there an easy way to check whether a representation for n is minimal?

Zeckendorf Representation. Every positive integer n can be uniquely expressed as the sum of distinct non-consecutive Fibonacci numbers.

5a. Show that every positive integer n has a Zeckendorf representation.

5b. Show that the representation is unique. (*Hint:* First show that the sum of distinct, non-consecutive Fibonacci numbers whose largest member is F_n is less than F_{n+1}.)

An Application

6. Can you think of a way to use the Zeckendorf representation to convert between miles and kilometers?

Negative Fibonacci Indices. We can extend the Fibonacci sequence to negative indices by applying the Fibonacci recursive formula "backwards". Every number can be uniquely represented as a sum of distinct non-consecutive negatively indexed Fibonacci numbers.

7a. Let $F_0 = 0$ and $F_1 = 1$. Find the values of F_{-1} to F_{-10}.

7b. Use only the negatively indexed Fibonacci numbers to represent the integers from -10 to 10.

Another Representation. Every integer can be written uniquely as a sum of Fibonacci numbers and their additive inverses such that every two terms of the same sign differ in index by at least 4 and every two terms of different sign differ in index by at least 3. This is called the *far-difference representation*. For example:

$$1 = 1 \qquad 5 = 5$$
$$2 = 2 \qquad 6 = 8 - 2$$
$$3 = 3 \qquad 7 = 8 - 1$$
$$4 = 5 - 1 \qquad 8 = 8$$

8. Find the far-difference representations for the numbers 9 to 18 and the numbers 70 and 127.

References

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